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Some Radius Problems Related to a Certain Subclass of Analytic Functions

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Abstract For real parameters α and β such that $0 \leq \alpha < 1 < \beta$, we denote by $\mathcal{S}(\alpha, \beta)$ the class of normalized analytic functions which satisfy the following two-sided inequality:

$$\alpha < \Re\left(\frac{zf'(z)}{f(z)}\right) < \beta, \quad z \in \mathbb{U},$$

where $\mathbb{U}$ denotes the open unit disk. We find a sufficient condition for functions to be in the class $\mathcal{S}(\alpha, \beta)$ and solve several radius problems related to other well-known function classes.

Keywords Analytic functions, univalent functions, starlike functions, functions of bounded real positive real part, radius problems

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1 Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$, analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\},$$

which are normalized by

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ composed of functions which are univalent in $\mathbb{U}$. As usual, we denote by $\mathcal{S}^*$ and $\mathcal{K}$ the classes of functions in $\mathcal{A}$ which are, respectively, starlike and convex in $\mathbb{U}$. It is well known that

$$\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}.$$

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We say that f is subordinate to F in $\mathbb{U}$, written as $f \prec F$ $(z \in \mathbb{U})$, if and only if

$$f(z) = F\big(w(z)\big)$$

for some Schwartz function $w(z)$ such that

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1, \quad z \in \mathbb{U}.$$

If F is univalent in $\mathbb{U}$, then the subordination $f \prec F$ is equivalent to

$$f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

We denote by $\mathcal{S}^*(A, B)$ the subclass of $\mathcal{S}^*$ consisting of the functions in $\mathcal{A}$ such that

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U}.$$

The subclass $\mathcal{SP}$ of the function class $\mathcal{A}$ is composed of parabolic starlike functions in $\mathbb{U}$, which satisfy the following inequality (see [9]):

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \Re\left(\frac{zf'(z)}{f(z)}\right), \quad z \in \mathbb{U}.$$

Recently, Sokół [2, 10, 11] introduced the class $\mathcal{SL}$ as a subclass of $\mathcal{S}^*$, which consists of functions $f(z)$ in $\mathcal{A}$ such that

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}, \quad z \in \mathbb{U}.$$

Moreover, a function $f \in \mathcal{A}$ is said to be strongly starlike of order α $(0 \leq \alpha < 1)$ in $\mathbb{U}$ if

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| \leq \frac{\pi}{2}\alpha, \quad z \in \mathbb{U}.$$

Definition 1.1 *Let the parameters α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}(\alpha, \beta)$ if f satisfies the following inequality*:

$$\alpha < \Re\left(\frac{zf'(z)}{f(z)}\right) < \beta, \quad z \in \mathbb{U};\ 0 \leq \alpha < 1 < \beta.$$

We remark that, for given parameters α and β $(0 \leq \alpha < 1 < \beta)$, $f \in \mathcal{S}(\alpha, \beta)$ if and only if f satisfies each of the following two subordination relationships:

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z}, \quad z \in \mathbb{U} \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}, \quad z \in \mathbb{U}.$$

The above-defined function class $\mathcal{S}(\alpha, \beta)$ was introduced by Kuroki and Owa [5]. By using the following lemma, they also investigated several coefficient estimates for $f \in \mathcal{S}(\alpha, \beta)$.

Lemma 1.2 (Kuroki and Owa [5]) *Let $f(z) \in \mathcal{A}$ and $0 \leq \alpha < 1 < \beta$. Then $f \in \mathcal{S}(\alpha, \beta)$ if and only if*

$$\frac{zf'(z)}{f(z)} \prec 1 + \left(\frac{\beta-\alpha}{\pi}\right)\mathrm{i}\log\left(\frac{1-\mathrm{e}^{2\pi\mathrm{i}\left(\frac{1-\alpha}{\beta-\alpha}\right)}z}{1-z}\right), \quad z \in \mathbb{U}. \tag{1.1}$$

Lemma 1.2 means that the function $p(z)$ defined by

$$p(z) = 1 + \left(\frac{\beta-\alpha}{\pi}\right)\mathrm{i}\log\left(\frac{1-\mathrm{e}^{2\pi\mathrm{i}\left(\frac{1-\alpha}{\beta-\alpha}\right)}z}{1-z}\right) \tag{1.2}$$

maps the unit disk $\mathbb{U}$ onto the strip domain w with $\alpha < \Re(w) < \beta$. We also note that the function $f \in \mathcal{A}$, given by

$$f(z) = z \exp\left(\left(\frac{\beta-\alpha}{\pi}\right)\mathrm{i}\int_0^z \frac{1}{t}\log\left(\frac{1-\mathrm{e}^{2\pi\mathrm{i}\left(\frac{1-\alpha}{\beta-\alpha}\right)}z}{1-z}\right)dt\right) \tag{1.3}$$

is in the class $\mathcal{S}(\alpha,\beta)$.

In our present investigation, we first find a sufficient condition for functions to be in the class $\mathcal{S}(\alpha,\beta)$. We then solve several radius problems related to other well-known function classes. For various other radius problems, which were considered recently for many different analytic function classes, the interested reader may be referred (for example) to the works [1, 4, 8, 12].

2 Relations Involving Bounds on the Real Parts

Lemma 2.1 below is a fairly well-known result.

Lemma 2.1 (MacGregor [6]) *Let $f \in \mathcal{A}$. Also let*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathbb{U},\ 0 \leq \alpha < 1.$$

Then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \Phi(\alpha), \quad z \in \mathbb{U},$$

where

$$\Phi(\alpha) := \begin{cases} \dfrac{1-2\alpha}{2(2^{1-2\alpha}-1)}, & \alpha \neq \dfrac{1}{2}, \\ \dfrac{1}{2\log 2}, & \alpha = \dfrac{1}{2}. \end{cases} \tag{2.1}$$

Another known result (Lemma 2.2 below) will also be needed in finding the relations involving upper bounds.

Lemma 2.2 (Miller and Mocanu [7]) *Let Ξ be a set in the complex plane $\mathbb{C}$ and let b be a complex number such that $\Re(b) > 0$. Suppose that a function $\vartheta : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ satisfies the following condition:*

$$\vartheta(\mathrm{i}\rho,\sigma;z) \notin \Xi, \quad z \in \mathbb{U},\ \rho,\sigma \leq -\frac{|b-\mathrm{i}\rho|^2}{2\Re(b)}.$$

If the function $p(z)$ defined by

$$p(z) = b + b_1 z + b_2 z^2 + \cdots$$

is analytic in $\mathbb{U}$ and if

$$\vartheta\big(p(z), zp'(z); z\big) \in \Xi,$$

then

$$\Re\{p(z)\} > 0, \quad z \in \mathbb{U}.$$

Theorem 2.3 *Let $f \in \mathcal{A}$, $\beta > 1$ and*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) < \beta, \quad z \in \mathbb{U}. \tag{2.2}$$

Then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \Psi(\beta) := \frac{-1+2\beta+\sqrt{4\beta^2-4\beta+9}}{4}. \tag{2.3}$$

Proof First, we note that

$$\Psi(\beta) := \frac{-1+2\beta+\sqrt{4\beta^2-4\beta+9}}{4} > 1, \quad \beta > 1.$$

Thus, if we let

$$p(z) = \frac{1}{1-\Psi(\beta)}\left(\frac{zf'(z)}{f(z)} - \Psi(\beta)\right), \tag{2.4}$$

then $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. Differentiating both sides of (2.4) with respect to z, we easily obtain

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} &= [1-\Psi(\beta)]p(z) + \Psi(\beta) + \frac{[1-\Psi(\beta)]zp'(z)}{[1-\Psi(\beta)]p(z)+\Psi(\beta)} \\ &= \psi\big(p(z), zp'(z)\big), \end{aligned}$$

where

$$\psi(r,s) := [1-\Psi(\beta)]r + \Psi(\beta) + \frac{[1-\Psi(\beta)]s}{[1-\Psi(\beta)]r+\Psi(\beta)}.$$

Using (2.2), we have

$$\{\psi\big(p(z), zp'(z)\big) : z \in \mathbb{U}\} \subset \{w : w \in \mathbb{C} \text{ and } \Re(w) < \beta\} =: \Omega.$$

Now, for all real numbers $\rho, \sigma \leq -\frac{|1-\mathrm{i}\rho|^2}{2}$, we have

$$\begin{aligned} \Re\{\psi(i\rho,\sigma)\} &= \Re\left([1-\Psi(\beta)]\mathrm{i}\rho + \Psi(\beta) + \frac{[1-\Psi(\beta)]\sigma}{[1-\Psi(\beta)]\mathrm{i}\rho+\Psi(\beta)}\right) \\ &= \Psi(\beta) - [\Psi(\beta)-1]\frac{\sigma\Psi(\beta)}{[\Psi(\beta)]^2+[1-\Psi(\beta)]^2\rho^2} \\ &\geq \Psi(\beta) + \frac{\Psi(\beta)[\Psi(\beta)-1](1+\rho^2)}{2([\Psi(\beta)]^2+[1-\Psi(\beta)]^2\rho^2)}. \end{aligned}$$

If we let the function $g(\rho)$ be given by

$$g(\rho) = \frac{1+\rho^2}{[\Psi(\beta)]^2+[\Psi(\beta)-1]^2\rho^2},$$

then $g(\rho)$ is a continuous even function of the argument ρ and $g(\rho)$ satisfies each of the following relationships:

$$g(0) = \frac{1}{[\Psi(\beta)]^2}$$

and

$$\lim_{\rho\to\infty} g(\rho) = \frac{1}{[\Psi(\beta)-1]^2} > \frac{1}{[\Psi(\beta)]^2}.$$

Also, upon differentiating $g(\rho)$ with respect to ρ, we obtain

$$g'(\rho) = \frac{2[2\Psi(\beta)-1]\rho}{\{[\Psi(\beta)]^2+[\Psi(\beta)-1]^2\rho^2\}^2}.$$

Hence, $g'(\rho) = 0$ occurs only at $\rho = 0$. Therefore, we have

$$g(\rho) \geq \frac{1}{[\Psi(\beta)]^2}, \quad \rho \in \mathbb{R},$$

which yields

$$\begin{aligned}\Re\{\psi(\mathrm{i}\rho,\sigma)\} &\geq \Psi(\beta)+\frac{\Psi(\beta)[\Psi(\beta)-1]g(\rho)}{2}\\ &\geq \frac{2[\Psi(\beta)]^2+\Psi(\beta)-1}{2\Psi(\beta)}=\beta.\end{aligned}$$

This shows that $\Re\{\psi(\mathrm{i}\rho,\sigma)\} \notin \Omega$. By Lemma 2.2, we thus conclude that $\Re\{p(z)\} > 0$ and that the inequality (2.3) holds true. The proof of Theorem 2.3 is thus complete. $\square$

By combining Lemma 2.1 and Theorem 2.3, we can obtain the following result.

Theorem 2.4 *Let $f \in \mathcal{A}$. Suppose also that*

$$\alpha < \Re\left(1+\frac{zf''(z)}{f'(z)}\right) < \beta, \quad z \in \mathbb{U},\ 0 \leq \alpha < 1 < \beta. \tag{2.5}$$

Then

$$\Phi(\alpha) < \Re\left(\frac{zf'(z)}{f(z)}\right) < \Psi(\beta), \tag{2.6}$$

where $\Phi(\alpha)$ and $\Psi(\beta)$ are given in (2.1) *and* (2.3), *respectively.*

3 Radius Problems Involving Subclasses of Analytic Functions

Our first result on the radius problem involves the function class $\mathcal{S}(\alpha,\beta)$.

Theorem 3.1 *Let the function f be in the class $\mathcal{S}(\alpha,\beta)$. Then, for each z ($|z| = r < 1$),*

$$1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\left(a_1(r)\right) < \Re\left(\frac{zf'(z)}{f(z)}\right) < 1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\left(a_2(r)\right)$$

and

$$\left(\frac{\beta-\alpha}{\pi}\right)\log\left(t_1(r)\right) < \Im\left(\frac{zf'(z)}{f(z)}\right) < \left(\frac{\beta-\alpha}{\pi}\right)\log\left(t_2(r)\right),$$

where

$$a_1(r) := \frac{(r^2-r^4\cos\varphi)\sin\varphi+\sqrt{\mathfrak{D}(r)}}{(\sin^2\varphi-1)r^4+2r^2-1}, \tag{3.1}$$

$$a_2(r) := \frac{(r^2-r^4\cos\varphi)\sin\varphi-\sqrt{\mathfrak{D}(r)}}{(\sin^2\varphi-1)r^4+2r^2-1}, \tag{3.2}$$

$$t_1(r) := \frac{\sqrt{1-2r^2\cos\varphi+r^4}-\left(\sqrt{2(1-\cos\varphi)}\right)r}{1-r^2}, \tag{3.3}$$

$$t_2(r) := \frac{\sqrt{1-2r^2\cos\varphi+r^4}+\left(\sqrt{2(1-\cos\varphi)}\right)r}{1-r^2} \tag{3.4}$$

and

$$\begin{aligned}\mathfrak{D}(r) := {}& r^4(1-r^2\cos\varphi)^2\sin^2\varphi+r^2(1-\cos\varphi)[r^4(\sin^2\varphi-1)+2r^2-1]\\ &\cdot[r^2(1+\cos\varphi)-2],\end{aligned} \tag{3.5}$$

with φ being given by

$$\varphi := 2\left(\frac{1-\alpha}{\beta-\alpha}\right)\pi.$$

Proof Suppose that $f \in \mathcal{S}(\alpha,\beta)$. Then, by Lemma 1.2, we have

$$\frac{zf'(z)}{f(z)} \prec 1 + \left(\frac{\beta-\alpha}{\pi}\right)\mathrm{i}\log\left(\frac{1-\mathrm{e}^{2\pi\mathrm{i}(\frac{1-\alpha}{\beta-\alpha})}z}{1-z}\right), \quad z \in \mathbb{U}.$$

Thus, by the definition of subordination, there is a Schwartz function $w(z)$, satisfying the following conditions:

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1, \quad z \in \mathbb{U},$$

such that

$$\frac{zf'(z)}{f(z)} = 1 + \left(\frac{\beta-\alpha}{\pi}\right)\mathrm{i}\log\left(\frac{1-\mathrm{e}^{2\pi\mathrm{i}(\frac{1-\alpha}{\beta-\alpha})}w(z)}{1-w(z)}\right), \quad z \in \mathbb{U}.$$

We now put

$$q(z) = \frac{1-\mathrm{e}^{2\pi\mathrm{i}(\frac{1-\alpha}{\beta-\alpha})}w(z)}{1-w(z)},$$

which readily yields

$$q(z) - 1 = \big(q(z) - \mathrm{e}^{2\pi\mathrm{i}(\frac{1-\alpha}{\beta-\alpha})}\big)w(z).$$

For $|z| \leq r < 1$, using the known fact that (see [3])

$$|w(z)| \leq |z|, \quad z \in \mathbb{U},$$

we find that

$$|q(z) - 1| \leq |q(z) - \mathrm{e}^{2\pi\mathrm{i}(\frac{1-\alpha}{\beta-\alpha})}| \cdot r, \quad |z| \leq r < 1. \tag{3.6}$$

If we put

$$q(z) = u + \mathrm{i}v \quad \text{and} \quad \varphi = 2\pi\left(\frac{1-\alpha}{\beta-\alpha}\right),$$

then, upon squaring both sides of (3.6), we get

$$\left(u - \frac{1-r^2\cos\varphi}{1-r^2}\right)^2 + \left(v + \frac{r^2\sin\varphi}{1-r^2}\right)^2 \leq \frac{2r^2(1-\cos\varphi)}{(1-r^2)^2}. \tag{3.7}$$

Hence, q maps the disk

$$\mathbb{U}_r := \{z : z \in \mathbb{C} \text{ and } |z| \leq r < 1\}$$

onto the circle which the center C is given by

$$\mathrm{C} : \left(\frac{1-r^2\cos\varphi}{1-r^2}, -\frac{r^2\sin\varphi}{1-r^2}\right)$$

and radius R given by

$$R := \sqrt{2(1-\cos\varphi)}\left(\frac{r}{1-r^2}\right).$$

We note also that the origin O is outside of the circle (3.7).

We shall now find the bounds of $|q(z)|$. Since the origin O is outside of the circle (3.7), $|q(z)|$ is less than the sum of $\overline{\mathrm{OC}}$ and the radius R and $|q(z)|$ is greater than the difference of $\overline{\mathrm{OC}}$ and the radius R, that is,

$$|q(z)| \leq \frac{\sqrt{1-2r^2\cos\varphi+r^4} + \big(\sqrt{2(1-\cos\varphi)}\big)r}{1-r^2} =: t_2(r)$$

and

$$|q(z)| \geq \frac{\sqrt{1-2r^2\cos\varphi+r^4}-\big(\sqrt{2(1-\cos\varphi)}\big)r}{1-r^2} =: t_1(r),$$

which are already given by (3.4) and (3.3), respectively.

Next, in order to find the bounds of $\arg\{q(z)\}$, we let $v = au$ be the equation of a straight line L which is tangent to the circle (3.7). Then u satisfies the following equation:

$$(1+a^2)u^2 + 2\left(-\frac{1-r^2\cos\varphi}{1-r^2}+\frac{ar^2\sin\varphi}{1-r^2}\right)u + \frac{(1-r^2\cos\varphi)^2+r^4\sin^2\varphi-2r^2(1-\cos\varphi)}{(1-r^2)^2} = 0.$$

Since the line L is tangent to the circle (3.2), we have

$$\left(-\frac{1-r^2\cos\varphi}{1-r^2}+\frac{ar^2\sin\varphi}{1-r^2}\right)^2 - (1+a^2)\left(\frac{(1-r^2\cos\varphi)^2+r^4\sin^2\varphi-2r^2(1-\cos\varphi)}{(1-r^2)^2}\right) = 0.$$

Solving this last equation for the unknown parameter a, we can obtain precisely the solutions $a_1(r)$ and $a_2(r)$ asserted by the equations (3.1) and (3.2) in terms of $\mathfrak{D}$ given by (3.5). Therefore, the upper and the lower bounds of $\arg q(z)$ are $\arctan\big(a_1(r)\big)$ and $\arctan\big(a_2(r)\big)$, respectively. Hence, $\log(q(z))$ maps the circle $\mathbb{U}_r$ into the rectangle $\mathbb{D}_1$, where

$$\mathbb{D}_1 = \{(u,v) : \log\big(t_2(r)\big) \leq u \leq \log\big(t_1(r)\big) \text{ and } \arctan\big(a_2(r)\big) \leq v \leq \arctan\big(a_1(r)\big)\}.$$

Thus, clearly, the function $\mathrm{i}\log\big(q(z)\big)$ maps the circle $\mathbb{U}_r$ into the rectangle $\mathbb{D}_2$, where

$$\mathbb{D}_2 = \{(u,v) : -\arctan\big(a_1(r)\big) \leq u \leq -\arctan\big(a_2(r)\big) \text{ and } \log\big(t_2(r)\big) \leq v \leq \log\big(t_1(r)\big)\}.$$

Multiplying by $\frac{\beta-\alpha}{\pi}$ each bound of the rectangle $\mathbb{D}_2$ and translating the region by 1 along the u-axis, we can obtain the following region:

$$\mathbb{D} = \left\{(u,v) : 1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\big(a_1(r)\big) \leq u \leq 1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\big(a_2(r)\big) \text{ and } \left(\frac{\beta-\alpha}{\pi}\right)\log\big(t_2(r)\big) \leq v \leq \left(\frac{\beta-\alpha}{\pi}\right)\log\big(t_1(r)\big)\right\},$$

which is mapped into the circle $\mathbb{U}_r$ by the function $p(z)$ given by

$$p(z) = 1+\left(\frac{\beta-\alpha}{\pi}\right)\mathrm{i}\log\big(q(z)\big).$$ □

Theorem 3.2 *Let α, β, γ and δ be given such that*

$$0 \leq \alpha < \gamma < 1 \quad \textit{and} \quad \beta > \delta > 1.$$

Let the function f be in the class $\mathcal{S}(\alpha,\beta)$. Suppose also that $a_1(r)$ and $a_2(r)$ are given (as in Theorem 3.1) by (3.1) and (3.2), respectively. Then

$$f \in \mathcal{S}(\gamma,\delta), \quad |z| \leq r_0,$$

where

$$r_0 = \min\{r_1, r_2\}, \quad r_1, r_2 \in (0,1),$$

and r_1 and r_2 are the smallest root of the following equations:

$$1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\big(a_1(r)\big)-\gamma=0$$

and

$$1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\big(a_2(r)\big)-\delta=0,$$

respectively.

Proof By Theorem 3.1, for each z ($|z|=r$), the function f satisfies the following two-sided inequality:

$$1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\big(a_1(r)\big)<\Re\left(\frac{zf'(z)}{f(z)}\right)<1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\big(a_2(r)\big).$$

For the function f to be in the class $\mathcal{S}(\gamma,\delta)$, it suffices to satisfy the following inequalities:

$$1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\big(a_1(r)\big)>\gamma \tag{3.8}$$

and

$$1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\big(a_2(r)\big)<\delta. \tag{3.9}$$

We now define a function $g:[0,1]\to\mathbb{R}$ by

$$g(r):=1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\big(a_1(r)\big)-\gamma.$$

Then g is continuous and $g(0)=1-\gamma>0$. Since

$$\lim_{r\to 1-}a_1(r)=\frac{1-\cos\varphi}{\sin\varphi}\quad\text{and}\quad\tan^2\left(\frac{1}{2}\varphi\right)=\frac{1-\cos\varphi}{\sin\varphi}, \tag{3.10}$$

we have

$$\lim_{r\to 1-}g(r)=\alpha-\gamma<0.$$

Hence, there exists a solution of the equation $g(r)=0$ in $(0,1)$. Let $r_1\in(0,1)$ be the smallest root of $g(r)=0$. Then $g(r)>0$ for all $r<r_1$. Therefore,

$$1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\big(a_1(r)\big)>\gamma$$

for all $r<r_1$. Using the same argument as above, we can show that there exists a solution $r_2\in(0,1)$ of the equation:

$$1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\big(a_2(r)\big)-\delta=0$$

and that

$$1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\big(a_2(r)\big)<\delta$$

for all $r<r_2$. Hence, if we put $r_0=\min\{r_1,r_2\}$, then the function f satisfies (3.8) and (3.9). Consequently, $f\in\mathcal{S}(\gamma,\delta)$ in $|z|\le r_0$. □

Theorem 3.3 *Let $f \in \mathcal{S}(\alpha,\beta)$. Then the radius of f to be a strongly starlike function of order γ in $\mathbb{U}$ is r_0, where $r_0 \in (0,1)$ is the smallest root of the following equation:*

$$\arctan\left(\frac{(\frac{\beta-\alpha}{\pi})\log\big(t_2(r)\big)}{1-(\frac{\beta-\alpha}{\pi})\arctan\big(a_1(r)\big)}\right)-\frac{\pi}{2}\gamma=0, \tag{3.11}$$

where $a_1(r)$ and $t_2(r)$ are given (as in Theorem 3.1) *by* (3.1) *and* (3.4), *respectively.*

Proof We first note that

$$\log(t_2(r)) = -\log(t_1(r)).$$

Hence, by Theorem 3.1, for $f \in \mathcal{S}(\alpha,\beta)$, we have

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| \le \arctan\left(\frac{(\frac{\beta-\alpha}{\pi})\log\big(t_2(r)\big)}{1-(\frac{\beta-\alpha}{\pi})\arctan\big(a_1(r)\big)}\right).$$

Thus, for the function f to be a strongly starlike function of order γ in $\mathbb{U}$, it suffices to satisfy the following inequality:

$$h(r) := \arctan\left(\frac{(\frac{\beta-\alpha}{\pi})\log\big(t_2(r)\big)}{1-(\frac{\beta-\alpha}{\pi})\arctan\big(a_1(r)\big)}\right)-\frac{\pi}{2}\gamma<0.$$

Using these observations in (3.10), we can easily show that

$$h(0) = -\frac{\pi}{2}\gamma < 0 \quad \text{and} \quad \lim_{r\to 1-} h(r) = \infty.$$

Hence, there exists a solution of the equation $h(r) = 0$ in $(0,1)$. Let $r_0 \in (0,1)$ be the smallest root of the equation $h(r) = 0$. Then $h(r) < 0$ for $r < r_0$. Thus, f is a strongly starlike function of order γ for z $(|z| \le r_0)$. □

Putting $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$ and $\gamma = \frac{1}{2}$ in Theorem 3.3, we can obtain the following corollary.

Corollary 3.4 *Let $f \in \mathcal{S}(\frac{1}{2},\frac{3}{2})$. Then the radius of f to be a strongly starlike function of order $\frac{1}{2}$ in $\mathbb{U}$ is* $0.981868\cdots$.

Theorem 3.5 *Let $f \in \mathcal{S}(\alpha,\beta)$. Also let $a_1(r)$ and $t_2(r)$ be given (as in Theorem* 3.1) *by* (3.1) *and* (3.4), *respectively. Then the radius of f to be in the class $\mathcal{SP}$ is r_0, where $r_0 \in (0,1)$ is the smallest root of the following equation:*

$$\left(\frac{(\beta-\alpha)}{\pi}\right)^2\left[\log\big(t_2(r)\big)\right]^2+\left(\frac{2(\beta-\alpha)}{\pi}\right)\arctan\big(a_1(r)\big)-1=0. \tag{3.12}$$

Proof We note that $f \in \mathcal{SP}$ if and only if the function $\frac{zf'(z)}{f(z)}$ is in the parabolic region given by

$$\Lambda = \{(u,v) : v^2 < 2u-1\}.$$

Thus, for the function f to be in the class $\mathcal{SP}$, it suffices to show that the point

$$\left(1-\left[\frac{\beta-\alpha}{\pi}\right]\arctan\big(a_1(r)\big), \left[\frac{\beta-\alpha}{\pi}\right]\log\big(t_2(r)\big)\right)$$

is in the parabolic region Λ, that is,

$$\left[\left(\frac{\beta-\alpha}{\pi}\right)\log\big(t_2(r)\big)\right]^2 < 2\left[1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\big(a_1(r)\big)\right]-1.$$

We now define a function $k : [0,1] \to \mathbb{R}$ by

$$k(r) := \left(\frac{\beta-\alpha}{\pi}\right)^2 \left[\log\left(t_2(r)\right)\right]^2 + \left(\frac{2(\beta-\alpha)}{\pi}\right) \arctan\left(a_1(r)\right) - 1.$$

Then

$$k(0) = -1 < 0 \quad \text{and} \quad \lim_{r\to 1-} k(r) = \infty.$$

Hence, there exists a solution of the equation $k(r) = 0$ in $(0,1)$. Let $r_0 \in (0,1)$ be the smallest root of $k(r) = 0$. Then $k(r) < 0$ for all $r < r_0$. Hence, $f(z) \in \mathcal{SP}$ for all z $(|z| \le r_0)$. $\square$

Putting $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$ in Theorem 3.5, we can obtain the following corollary.

Corollary 3.6 *Let $f \in \mathcal{S}(\frac{1}{2}, \frac{3}{2})$. Then the radius of f to be in the class $\mathcal{SP}$ is* $0.697818\cdots$.

Theorem 3.7 *Let the function f be in the class $\mathcal{S}(\alpha,\beta)$. Suppose also that $a_1(r)$, $a_2(r)$, $t_1(r)$ and $t_2(r)$ are given (as in Theorem 3.1) by (3.1) to (3.4). Then*

$$f \in \mathcal{SL}, \quad |z| \le r_0,$$

where

$$r_0 := \min\{r_1, r_2\}, \quad r_1, r_2 \in (0,1),$$

and r_1 and r_2 are the smallest root of the following equations:

$$\begin{aligned}&\left(\left[1 - \left(\frac{\beta-\alpha}{\pi}\right) \arctan\left(a_1(r)\right)\right]^2 - \left[\left(\frac{\beta-\alpha}{\pi}\right) \log\left(t_2(r)\right)\right]^2 - 1\right)^2 \\ &\quad + \left(\frac{2(\beta-\alpha)}{\pi}\right)^2 \left[\log\left(t_1(r)\right)\right]^2 \cdot \left[1 - \left(\frac{\beta-\alpha}{\pi}\right) \arctan\left(a_1(r)\right)\right]^2 - 1 = 0, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned}&\left(\left[1 - \left(\frac{\beta-\alpha}{\pi}\right) \arctan\left(a_2(r)\right)\right]^2 - \left[\left(\frac{\beta-\alpha}{\pi}\right) \log\left(t_2(r)\right)\right]^2 - 1\right)^2 \\ &\quad + \left(\frac{2(\beta-\alpha)}{\pi}\right)^2 \left[\log\left(t_1(r)\right)\right]^2 \cdot \left[1 - \left(\frac{\beta-\alpha}{\pi}\right) \arctan\left(a_2(r)\right)\right]^2 - 1 = 0, \end{aligned} \tag{3.14}$$

respectively.

Proof We note that $f \in \mathcal{SL}$ if and only if the function $\frac{zf'(z)}{f(z)}$ is in the bounded region Γ given by

$$\Gamma := \{(u,v) : u^4 + v^4 + 1 + 2u^2v^2 - 2u^2 - 2v^2 < 1\}.$$

We note also that this region Γ is symmetric to the u-axis in uv-plane and

$$\log\left(t_1(r)\right) = -\log\left(t_2(r)\right).$$

Thus, if

$$\left(1 - \left[\frac{\beta-\alpha}{\pi}\right] \arctan\left(a_1(r)\right), \left[\frac{\beta-\alpha}{\pi}\right] \log\left(t_2(r)\right)\right) \in \Gamma \tag{3.15}$$

and

$$\left(1 - \left[\frac{\beta-\alpha}{\pi}\right] \arctan\left(a_2(r)\right), \left[\frac{\beta-\alpha}{\pi}\right] \log\left(t_2(r)\right)\right) \in \Gamma, \tag{3.16}$$

then $f \in \mathcal{SL}$ for $|z| = r < 1$. The conditions (3.15) and (3.16) are equivalent to the following inequalities:

$$\left(\left[1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\left(a_1(r)\right)\right]^2-\left[\left(\frac{\beta-\alpha}{\pi}\right)\log\left(t_2(r)\right)\right]^2-1\right)^2 \\ +\left(\frac{2(\beta-\alpha)}{\pi}\right)^2\left[\log\left(t_1(r)\right)\right]^2\cdot\left[1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\left(a_1(r)\right)\right]^2-1<0 \tag{3.17}$$

and

$$\left(\left[1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\left(a_2(r)\right)\right]^2-\left[\left(\frac{\beta-\alpha}{\pi}\right)\log\left(t_2(r)\right)\right]^2-1\right)^2 \\ +\left(\frac{2(\beta-\alpha)}{\pi}\right)^2\left[\log\left(t_1(r)\right)\right]^2\cdot\left[1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\left(a_2(r)\right)\right]^2-1<0, \tag{3.18}$$

respectively. We now define a function $g : [0, 1] \to \mathbb{R}$ by

$$g(r)=\left(\left[1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\left(a_1(r)\right)\right]^2-\left[\left(\frac{\beta-\alpha}{\pi}\right)\log\left(t_2(r)\right)\right]^2-1\right)^2 \\ +\left(\frac{2(\beta-\alpha)}{\pi}\right)^2\left[\log\left(t_1(r)\right)\right]^2\cdot\left[1-\left(\frac{\beta-\alpha}{\pi}\right)\arctan\left(a_1(r)\right)\right]^2-1.$$

Then g is continuous in $[0, 1]$. Furthermore, we have

$$g(0) = -1 \quad \text{and} \quad \lim_{r\to 1-} g(r) = \infty.$$

Hence, there exists a solution of the equation $g(r) = 0$ in $(0, 1)$. Let $r_1 \in (0, 1)$ be the smallest root of $g(r) = 0$. Then $g(r) < 0$ for all $r < r_1$. Hence, (3.17) holds true for all $r < r_1$. Using the same argument as above, we can find $r_2 \in (0, 1)$ such that (3.14) holds true and that, for all $r < r_2$, (3.18) holds true. Thus, if we put $r_0 = \min\{r_1, r_2\}$, then the function f satisfies (3.17) and (3.18). Consequently, $f \in \mathcal{SL}$ in $|z| \le r_0$. □

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